A LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE FOR CERTAIN HOMOTOPY FIXED POINT SPECTRA

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ABSTRACT. Let H and K be closed subgroups of the extended Morava stabilizer group G_n and suppose that H is normal in K. We construct a strongly convergent spectral sequence

$$H_c^*(K/H, (E_n^{hH})^* X) \Rightarrow (E_n^{hK})^* X,$$

where E_n^{hH} and E_n^{hK} are the continuous homotopy fixed point spectra of Devinatz and Hopkins. This spectral sequence turns out to be an Adams spectral sequence in the category of $K(n)_*$-local E_n^{hK}-modules.

INTRODUCTION

Suppose that Z is a spectrum and that G is a group acting on Z in some good point-set category of spectra. If G is, say, profinite, one might hope to construct, for each closed subgroup H of G, a "continuous homotopy fixed point spectrum", denoted (abusively) by Z^{hH}. There should be a "continuous homotopy fixed point spectral sequence"

$$H_c^*(H, Z^* X) \Rightarrow [X, Z^{hH}]^*$$

for all CW-spectra X, where $H_c^*(H, ?)$ denotes the continuous cohomology of the profinite group H. This of course requires that $Z^* X$ has an appropriate topology on which G acts continuously (see [19] or [6, Remark 1.7]). In addition, Z^{hH} should have the expected functorial properties and should agree with the ordinary homotopy fixed point spectrum when H is discrete.

We are interested in one very important example where such a construction is possible—namely, the action of the Morava stabilizer group G_n on the Landweber exact spectrum E_n. First recall the definitions. For a fixed prime number p, the coefficient ring E_{n*} is $W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$, where $|u_i| = 0$, $|u| = -2$, and as usual $W\mathbb{F}_{p^n}$ denotes the ring of Witt vectors with coefficients in the field $\mathbb{F}_{p^n}$ of p^n elements. $G_n = S_n \rtimes \mathrm{Gal}$, where S_n denotes the automorphism group of the height n Honda formal group law over $\mathbb{F}_{p^n}$, and $\mathrm{Gal} \equiv \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ denotes the Galois group of the field extension $\mathbb{F}_{p^n}/\mathbb{F}_p$. Morava's theory implies that G_n acts on E_n by ring spectrum maps in the stable category, and technology developed by

Goerss, Hopkins, and Miller (see [9], [10], [11], [12], [16]) then implies that E_n has a model on which G_n acts before passage to the stable category, so that homotopy fixed point spectra may be formed. Moreover, G_n is a profinite group—even a p-adic analytic group—and we constructed in [6] good continuous homotopy fixed point spectra for the action of this group on E_n. These *continuous* homotopy fixed point spectra are the homotopically significant spectra in this situation, since, for example, the continuous homotopy fixed point spectral sequences are generalized Adams spectral sequences and $E_n^{hG_n} = L_{K(n)}S^0$, the $K(n)_*$-localization of S^0.

Returning to the general situation, naturality of the continuous homotopy fixed point spectrum Z^{hH} implies that it is acted upon by the group $N(H)/H$. If $F = K/H$ is a finite subgroup of $N(H)/H$, we can form the ordinary homotopy F fixed point spectrum $(Z^{hH})^{hF}$, and we would expect this spectrum to be equivalent to the continuous homotopy fixed point spectrum Z^{hK}. This was proved for the action of G_n on E_n in [6]. If F is closed but not finite, the situation becomes more problematic. Here the relevant spectrum is the continuous homotopy F fixed points of E_n^{hH}. However, we do not have an intrinsic construction of such a spectrum—our construction of continuous homotopy fixed point spectra was specific to the action of G_n on E_n. Yet it is clear that $(E_n^{hH})^{hF}$ ought to be just E_n^{hK}. We can then ask whether there is a convergent spectral sequence

(0.1) $$H_c^*(K/H, (E_n^{hH})^* X) \Rightarrow [X, E_n^{hK}]^*;$$

such a spectral sequence provides a way of using information about E_n^{hH} to gain information about E_n^{hK}.

In this paper, we construct the spectral sequence (0.1) and prove that it is always strongly convergent. Our construction makes use of the highly structured results of [6]: if G is a closed subgroup of G_n, E_n^{hG} is a commutative S^0-algebra in the sense of [8] and the maps between these continuous homotopy fixed point spectra arising from functoriality are maps of commutative S^0-algebras. In particular, the inclusion $H \to K$ makes E_n^{hH} a commutative E_n^{hK}-algebra. We can then form a $K(n)_*$-local E_n^{hH}-Adams resolution of E_n^{hK} in the stable category of E_n^{hK}-modules, again in the sense of [8]. By neglect of structure, this Adams resolution gives us a diagram of cofibrations in the stable category. Map the CW-spectrum X into this diagram to obtain the desired spectral sequence.

This paper is organized as follows. In §1, we recall the definition and properties of the stable category of R-modules, for R a commutative S^0-algebra, and deal with some subtleties which will arise later. In §2, we give a general discussion of Adams spectral sequences in the stable category of R-modules. This material is a straightforward adaptation of work of Miller [15] and Bousfield [3]; these Adams spectral sequences have also been considered by Baker and Lazerev [1]. In §3, we identify the E_2-term of the aforementioned Adams spectral sequence with the continuous cohomology of K/H. Actually, we prove a more general result valid for E_n^{hK}-modules X; this is stated as Theorem 3.1. Along the way, we identify the algebra

$$\pi_* L_{K(n)}(E_n^{hH} \wedge_{E_n^{hK}} E_n^{hH})$$

of homology cooperations. The strong convergence of this spectral sequence is established in §4, and finally, in the Appendix, we prove that, if K/H is finite, the spectral sequence we have constructed agrees with the ordinary homotopy fixed point spectral sequence.

1. CATEGORIES OF MODULE SPECTRA

We use the framework of [8] in this paper. Our basic category of spectra is thus the category of S^0-modules $\mathcal{M}_{S^0}$; this category becomes the usual stable category, denoted $\mathcal{D}_{S^0}$, upon taking its homotopy category and inverting the weak equivalences. The advantage of the category of S^0-modules is that it is symmetric monoidal; that is, there is an associative, commutative, and unital smash product, denoted $\wedge$, and so one has the notion of S^0-algebras or commutative S^0-algebras. (These correspond to the earlier notions of A_∞ and E_∞ spectra, respectively.) If R is an S^0-algebra, there is then the evident notion of a (left) R-module, and one can form the category $\mathcal{M}_R$ of such R-modules and R-module maps between them. If R is commutative, a left R-module is the same as a right R-module, and one can define the smash product $M \wedge_R N$ of two R-modules to be the R-module given by the coequalizer diagram

$$M \wedge R \wedge N \underset{M \wedge \mu_N}{\overset{\mu_M \wedge N}{\rightrightarrows}} M \wedge N \longrightarrow M \wedge_R N,$$

where μ_M and μ_N are the module structure maps for M and N. This smash product makes $\mathcal{M}_R$ a symmetric monoidal category with R as the unit. The stable category $\mathcal{D}_R$ of R-modules, called the derived category in [8], is formed from the homotopy category of $\mathcal{M}_R$ by inverting the weak equivalences; i.e., those maps of R-modules, which, regarded as maps of spectra, induce isomorphisms of homotopy groups. Equivalently, $\mathcal{D}_R$ is the homotopy category of cell R-modules, again as defined in [8]. The categories $\mathcal{M}_R$ and $\mathcal{D}_R$ have the same formal properties as the categories $\mathcal{M}_{S^0}$ and $\mathcal{D}_{S^0}$, and we shall use these for the most part without comment. However, the (derived) smash product in $\mathcal{D}_R$ will be the source of some complications later, and thus we give a more detailed discussion now.

If M and N are in $\mathcal{M}_R$, the derived smash product $M \triangle_R N$ in $\mathcal{D}_R$ is defined to be $\Gamma M \wedge_R \Gamma N$, where ΓM and ΓN are cell R-modules weakly equivalent to M and N in $\mathcal{M}_R$. Of course, the canonical map $M \triangle_R N \to M \wedge_R N$ need not be an equivalence in $\mathcal{D}_R$. If $f : R \to R'$ is a map of commutative S^0-algebras, then there is an evident functor $f^* : \mathcal{M}_{R'} \to \mathcal{M}_R$. Since f^* preserves weak equivalences, we also have $f^* : \mathcal{D}_{R'} \to \mathcal{D}_R$ and a natural transformation $f^* M \triangle_R f^* N \to f^*(M \triangle_{R'} N)$ in $\mathcal{D}_R$ for M and N in $\mathcal{M}_{R'}$.

There are, however, certain situations where the map $M \triangle_R N \to M \wedge_R N$ is an equivalence. These will prove quite convenient in what follows.

Proposition 1.1 ([8, III, Theorem 3.8]). *If M is a cell R-module and $\phi : N \to N'$ is a weak equivalence in $\mathcal{M}_R$, then $M \wedge_R \phi : M \wedge_R N \to M \wedge_R N'$ is a weak equivalence. In particular, $M \wedge_R N \simeq M \triangle_R N$ whenever M is a cell R-module.*

To state the next condition, we need a little more preparation. Recall that the category of commutative R-algebras has the structure of a model category (see [8, VII]); we define a q-cofibrant commutative R-algebra to be a cofibrant object in this category. There is also the notion of a cell commutative R-algebra ([8, VII, Definition 4.11]), and any such object is q-cofibrant. These cell objects, however, do not in general have the homotopy type of cell R-modules.

Let $\overline{\mathcal{E}}_R$ denote the class of R-modules defined in [8, VII, 6]. Included in $\overline{\mathcal{E}}_R$ are all R-modules having the homotopy type of cell R-modules. The main reasons for interest in this class are the following results.

Theorem 1.2 ([8, VII, Theorem 6.7]). *Suppose that R is a q-cofibrant commutative S^0-algebra and that $M_i \in \overline{\mathcal{E}}_R$ for $i = 1, \ldots, k$. Then*

$$M_1 \triangle_R \cdots \triangle_R M_k \to M_1 \wedge_R \cdots \wedge_R M_k$$

is an equivalence in $\mathcal{D}_R$.

Theorem 1.3 ([8, VII, Theorem 6.5]). *Suppose that A is a q-cofibrant commutative R-algebra. Then A is in $\overline{\mathcal{E}}_R$.*

Since $\overline{\mathcal{E}}_R$ is closed under homotopy equivalences, finite $\wedge_R$-products, pushouts along cofibrations, and colimits of (countable) sequences of cofibrations, the next result is an easy consequence of Theorem 1.3.

Proposition 1.4. *Let R' be a q-cofibrant commutative R-algebra, and suppose the R-module M has the homotopy type of a cell R'-module (regarded as R-modules). Then $M \in \overline{\mathcal{E}}_R$.*

These results will all be relevant to our work, since, in [6], we constructed the continuous homotopy fixed point spectra E_n^{hG} to be cell commutative S^0-algebras.

We will also need some results from [8] on the Bousfield localization of R-modules. Suppose then that F is a cell R-module. A map $f : X \to Y$ of R-modules is an F_*^R-equivalence if $F \wedge f : F \wedge_R X \to F \wedge_R Y$ is a weak equivalence, and an R-module W is F_*^R-acyclic if $F \wedge_R W$ is weakly contractible. (We assume F is a cell R-module so that we can work with the ordinary, as opposed to the derived, smash product.) An R-module Y is F_*^R-local if $[W, Y]_*^R$ is trivial whenever W is F_*^R-acyclic. Here $[W, Y]_*^R$ denotes the group of maps $\Sigma^i W \to Y$ in $\mathcal{D}_R$; we will also denote this group by $[W, Y]_R^{-i}$. Finally, $f : X \to Y$ is called the F_*^R-localization of X if f is an F_*^R-equivalence and Y is F_*^R-local. If such a localization exists, it is unique up to canonical isomorphism in $\mathcal{D}_R$. It is proved in [8, VIII, 1] that X always has an F_*^R-localization, which we denote $\lambda : X \to L_F^R X$. Moreover, if R is a q-cofibrant S^0-algebra, e.g., $R = S^0$, and X is a cell commutative R-algebra, then λ can be constructed to be a map of cell commutative R-algebras and even to be natural in this category; i.e., the full subcategory of the category of commutative R-algebras and R-algebra maps whose objects are the cell commutative R-algebras.

If K is a cell S^0-module, and $F = R \wedge K$ as R-modules, then a map $f : X \to Y$ of R-modules is an F_*^R-equivalence if and only if it is a K_*-equivalence when regarded as a map of S^0-modules. Hence there is a canonical map $L_F^R X \to L_K X$ in $\mathcal{D}_{S^0}$, where we write $L_K X$ for $L_K^{S^0} X$. Furthermore, an S^0-module W is K_*-acyclic if and only if $R \wedge W$ is F_*^R-acyclic; this implies that $L_F^R X$ is K_*-local and therefore that $L_F^R X \to L_K X$ is a weak equivalence. In particular, K_*-localization defines a functor and natural transformation on $\mathcal{D}_R$, and the K_*-localization of a commutative R-algebra can be taken to be a commutative R-algebra (for R a q-cofibrant commutative S^0-algebra). The case with $K = K(n)$ and $R = E_n^{hG}$, for G a closed subgroup of G_n, will be the case of interest in this paper.

2. Adams spectral sequences in categories of module spectra

In this section we set up and discuss the convergence of Adams spectral sequences in localized categories of R-modules. Since we will be exclusively working in (full subcategories of) $\mathcal{D}_R$, we will, *in this section only*, write $\wedge_R$—instead of using the notation $\triangle_R$ introduced in §1—for the derived smash product.

Say that an R-module E is a *commutative homotopy R-algebra* if it is provided with maps $\eta : R \to E$ and $\mu : E \wedge_R E \to E$ (in $\mathcal{D}_R$) such that the expected diagrams commute. Let F be an R-module. Following Miller [15], we construct the F_*^R-local E-Adams resolution of an (F_*^R-local) R-module Y as an injective resolution of Y for an appropriate injective class.

Definition 2.1. With notation as above, an R-module is F_*^R-*local E-injective* if it is a retract of $L_F^R(Z \wedge_R E)$ for some R-module Z. Such an R-module will also be called E-*injective in* $L_F^R \mathcal{D}_R$, where $L_F^R \mathcal{D}_R$ is the category of F_*^R-local R-modules. A sequence of F_*^R-local R-modules $X' \to X \to X''$ is E-*exact in* $L_F^R \mathcal{D}_R$ if

$$[X', I]_*^R \longleftarrow [X, I]_*^R \longleftarrow [X'', I]_*^R$$

is exact for every F_*^R-local E-injective R-module I.

One can easily check that this defines an injective class in the sense of [14] in the category $L_F^R \mathcal{D}_R$. Thus each object Y in $L_F^R \mathcal{D}_R$ has a resolution by F_*^R-local E-injective R-modules, unique up to chain homotopy.

If

$$(2.1) \qquad * \longrightarrow Y \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

is such a resolution, then we may construct a diagram

$$(2.2) \qquad Y = Y^0 \xleftarrow{\;\;i\;\;} Y^1 \xleftarrow{\;\;i\;\;} Y^2 \longleftarrow \cdots$$

of exact triangles; in this diagram the map

$$L_F^R(j \wedge E) : L_F^R(Y^i \wedge_R E) \to L_F^R(I^i \wedge_R E)$$

is a split monomorphism. Conversely, a diagram of exact triangles as in (2.2) with each I^j F_*^R-local E-injective and $L_F^R(j \wedge_R E)$ split monic yields an injective resolution (2.1). Such a diagram is called an F_*^R-*local E-Adams resolution of Y* or an E-*Adams resolution of Y in* $L_F^R \mathcal{D}_R$ and is functorial up to chain homotopy.

By mapping an R-module X into an F_*^R-local E-Adams resolution of Y, we obtain a spectral sequence, called the F_*^R-*local E-Adams spectral sequence*, with

$$E_1^{s,t} = [X, I^s]_{t-s}^R.$$

(This indexing follows the convention that i and j are maps of degree 0, and k is a map of degree -1.) When R and F are understood, we will write this spectral sequence as $E_r^{s,t}(X, Y; E)$. Naturally, we hope that this spectral sequence converges strongly to Y; the rest of this section is devoted to establishing a criterion which guarantees this.

Following Bousfield [3, Definition 3.7], we define the class of F_*^R-*local E-nilpotent R-modules* to be the smallest class $\mathcal{C}$ of objects in $L_F^R \mathcal{D}_R$ such that

    i. $L_F^R E \in \mathcal{C}$.
    ii. $L_F^R(N \wedge_R X) \in \mathcal{C}$ whenever $N \in \mathcal{C}$ and $X \in \mathcal{D}_R$.
    iii. If $X \to Y \to Z \to \Sigma X$ is a cofiber sequence in $L_F^R \mathcal{D}_R$ and two of X, Y, Z are in $\mathcal{C}$, then so is the third.
    iv. $\mathcal{C}$ is closed under retracts in $L_F^R \mathcal{D}_R$.

Next, say that $Z \in L_F^R \mathcal{D}_R$ is L_F^R-E_*^R-*local* if $[W, Z]^R$ is trivial whenever $L_F^R(E \wedge_R W) \simeq *$. Observe that this is the same as saying that Z is $(F \wedge_R E)_*^R$-local. The following result is proved just as in [3, Lemma 3.8].

Proposition 2.2. *If Z is F_*^R-local E-nilpotent, then Z is L_F^R-E_*^R-local.*

The notion of an F_*^R-local E-nilpotent resolution of an object $Y \in L_F^R \mathcal{D}_R$ may also be defined along the lines of [3], and its uniqueness in an appropriate "pro-category" may also be established.

These resolutions are related to Adams spectral sequences as follows. Consider the Adams resolution (2.2), and let Y_s be the cofiber of $i^{(s+1)} : Y^{s+1} \to Y$. We can then construct commutative diagrams

$$
\begin{array}{ccccccccc}
Y^{s+1} & \longrightarrow & Y & \longrightarrow & Y_s & \longrightarrow & \Sigma Y^{s+1} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle i} & & \| & & \downarrow & & \downarrow{\scriptstyle i} & & \\
Y^s & \longrightarrow & Y & \longrightarrow & Y_{s-1} & \longrightarrow & \Sigma Y^s & \longrightarrow & \cdots
\end{array}
$$

of cofibration sequences, as well as a diagram

$$
(2.3) \qquad * = Y_{-1} \longleftarrow Y_0 \longleftarrow Y_1 \longleftarrow \cdots
$$
$$
\searrow \quad \nearrow \quad \searrow \quad \nearrow
$$
$$
\Sigma I^0 \qquad\qquad \Sigma I^1
$$

of exact triangles in $\mathcal{D}_R$. The spectral sequence obtained by mapping an R-module X into this diagram yields the F_*^R-local E-Adams spectral sequence. Moreover, the tower $\{Y_s\}_{s \geq 0}$ under Y is an F_*^R-local E-nilpotent resolution of Y (see [3, Lemma 5.7]). Thus, if Y is itself F_*^R-local E-nilpotent, then the constant tower $\{Y\}$ is pro-isomorphic to $\{Y_s\}$. This implies the following convergence result (see [3, Theorem 6.10]).

Proposition 2.3. *If Y is F_*^R-local E-nilpotent, then the F_*^R-local E-Adams spectral sequence converges strongly and conditionally to $[X, Y]_*^R$, for any R-module X. In addition, there exists s_0 such that $E_\infty^{s,*}(X, Y; E) = 0$ for all $s > s_0$ and any R-module X.*

Remark 2.4. We are using the notions of strong and conditional convergence as given in Boardman [2] applied to the unrolled exact couple

$$
[X, Y]_*^R = [X, Y^0] \longleftarrow [X, Y^1]_*^R \longleftarrow [X, Y^2]_*^R \longleftarrow \cdots
$$
$$
\searrow \quad \nearrow \qquad\qquad \searrow \quad \nearrow
$$
$$
[X, I^0]_*^R \qquad\qquad\qquad [X, I^1]_*^R
$$

Note that in the presence of conditional (resp. strong) convergence, strong (resp. conditional) convergence is equivalent to the condition that $\lim_{\leftarrow r}^1 E_r^{s,t}(X, Y; E) = 0$ for all s, t ([2, Theorem 7.3]).

The next result is a partial converse to Proposition 2.3 and will be useful to us later.

Proposition 2.5. *Suppose that Y is L_F^R-E_*^R-local and that there exist s_0 and r_0 such that for all R-modules X, the F_*^R-local E-Adams spectral sequence satisfies $E_{r_0}^{s,*}(X, Y; E) = 0$ whenever $s > s_0$. Then Y is F_*^R-local E-nilpotent.*

Proof. Since $\lim_{\leftarrow r}^1 E_r^{s,t}(X, Y; E) = 0$ for all s, t, it follows from [2, Theorem 7.4], applied to the unrolled exact couple

$$0 = [X, Y_{-1}]_*^R \longleftarrow [X, Y_0]_*^R \longleftarrow [X, Y_1]_*^R \longleftarrow \cdots$$
$$[X, I^0]_*^R \qquad\qquad [X, I^1]_*^R$$

that

    i. $\lim_{\leftarrow s}^1 [X, Y_s]_*^R = 0$ for all R-modules X,

    ii. $F_{s,t}(X, Y; E)/F_{s+1,t+1}(X, Y; E) \xrightarrow{\approx} E_\infty^{s,t}(X, Y; E)$,

where

$$F_{s,t}(X, Y; E) = \ker(\lim_{\leftarrow i}[X, Y_i]_{t-s}^R \longrightarrow [X, Y_{s-1}]_{t-s}^R)$$
$$= \ker([X, \operatorname*{holim}_{\leftarrow i} Y_i]_{t-s}^R \longrightarrow [X, Y_{s-1}]_{t-s}^R).$$

The horizontal vanishing line now implies that $F_{s_0+1,*}(X, Y; E) = 0$. Since X is arbitrary, this in turn implies that the map $p : \operatorname*{holim}_{\leftarrow i} Y_i \to Y_{s_0}$ is the inclusion of a summand in $\mathcal{D}_R$; in particular, $\operatorname*{holim}_{\leftarrow i} Y_i$ is F_*^R-local E-nilpotent.

Now consider the map $\iota : Y \to \operatorname*{holim}_{\leftarrow s} Y_s$ which lifts the canonical maps $Y \to Y_s$. (This map is unique since $\lim_{\leftarrow}^1 [Y, Y_s]_*^R = 0$.) ι is a map between L_F^R-E_*^R local objects; it is therefore an equivalence if and only if

$$L_F^R(E \wedge_R \iota) : L_F^R(E \wedge_R Y) \to L_F^R(E \wedge_R \operatorname*{holim}_{\leftarrow s} Y_s)$$

is an equivalence. We will prove that $L_F^R(E \wedge_R \iota)$ is an equivalence, and hence Y is F_*^R-local E-nilpotent.

Begin by observing that $L_F^R(E \wedge_R Y)$ is F_*-local E-injective, so that

$$E_\infty^{s,t}(X, L_F^R(E \wedge_R Y); E) = E_2^{s,t} = \begin{cases} [X, L_F^R(E \wedge_R Y)]_t^R & s = 0, \\ 0 & s \neq 0. \end{cases}$$

Since $L_F^R(E \wedge_R ?)$ applied to an F_*^R-local E-Adams resolution of Y is an F_*^R-local E-Adams resolution of $L_F^R(E \wedge_R Y)$, it follows as above that $\lim_{\leftarrow s}^1 [X, L_F^R(E \wedge_R Y_s)]_*^R = 0$ for all X. Hence there is a canonical map

$$L_F^R(E \wedge_R Y) \to \operatorname*{holim}_{\leftarrow s} L_F^R(E \wedge_R Y_s),$$

and this map is an equivalence.

Now there is also a canonical map

$$j : L_F^R(E \wedge_R \operatorname*{holim}_{\leftarrow s} Y_s) \to \operatorname*{holim}_{\leftarrow s} L_F^R(E \wedge_R Y_s),$$

whence the composition

$$L_F(E \wedge_R Y) \xrightarrow{L_F^R(E \wedge \iota)} L_F^R(E \wedge_R \operatorname*{holim}_{\leftarrow s} Y_s) \xrightarrow{j} \operatorname*{holim}_{\leftarrow s} L_F^R(E \wedge_R Y_s)$$

is an equivalence. Thus $\pi_* j$ is an epimorphism.

On the other hand, choose $r_0 : Y_{s_0} \to \operatorname*{holim}_{\leftarrow s} Y_s$ so that $r_0 \circ p = id$, and let r be the composition

$$\operatorname*{holim}_{\leftarrow s} L_F^R(E \wedge_R Y_s) \to L_F^R(E \wedge_R Y_{s_0}) \xrightarrow{L_F^R(E \wedge r_0)} L_F^R(E \wedge_R \operatorname*{holim}_{\leftarrow s} Y_s).$$

Then $r \circ j$ is the identity, so $\pi_* j$ is a monomorphism and therefore an isomorphism. This implies that $L_F^R(E \wedge_R \iota)$ is an equivalence, completing the proof. $\qquad\square$

3. IDENTIFICATION OF THE E_2-TERM

For the rest of this paper, the integer $n \geq 1$ will be fixed, so we will delete the subscript n from E_n or any of its continuous homotopy fixed point spectra. We will also write $L_{K(n)}$ as $\widehat{L}$, but will retain the notation $K(n)$, reserving the notation K for closed subgroups of G_n.

Let H and K be closed subgroups of G_n with H normal in K. The main result of [6] constructs $K(n)_*$-local cell commutative S^0-algebras E^{hH} and E^{hK} which are to be interpreted as continuous homotopy fixed point spectra. The inclusion of H in K induces a map $E^{hK} \to E^{hH}$ of commutative S^0-algebras, so that E^{hH} is a commutative E^{hK}-algebra. Moreover, there is an action of K/H on E^{hH} by E^{hK}-algebra maps, whose construction we now recall.

If

(3.1) $$G_n = U_0 \supsetneq U_1 \supsetneq U_2 \supsetneq \cdots \supsetneq U_i \supsetneq \cdots$$

is a sequence of open normal subgroups of G_n with $\bigcap_i U_i = \{e\}$, then

$$E^{hG} = \widehat{L}(\operatorname*{holim}_{\to i} E^{h(U_i G)})$$

for any closed subgroup G of G_n. Each left G_n-map $\rho : G_n/HU_i \to G_n/HU_i$ making the diagram

$$
\begin{array}{ccc}
G_n/HU_i & \xrightarrow{\quad\rho\quad} & G_n/HU_i \\
 & \searrow \qquad \swarrow & \\
 & G_n/KU_i &
\end{array}
$$

commute induces an $E^{h(KU_i)}$-algebra map $\mathbf{F}(\rho) : E^{h(U_i H)} \to E^{h(U_i H)}$ [6, Theorem 1]. If gH is an element of K/H, define such an automorphism of G_n/HU_i by sending xHU_i to $xgHU_i$. Then pass to homotopy colimits to obtain an E^{hK}-algebra automorphism of E^{hH}. Under the isomorphism

$$\pi_* \widehat{L}(E^{hH} \wedge E) \xrightarrow{\approx} \operatorname{Map}_c(H\backslash G_n, E_*)$$

of $\pi_* \widehat{L}(E \wedge E) \approx \operatorname{Map}_c(G_n, E_*)$-comodules ([6, Theorem 2])—here $H\backslash G_n$ denotes the space of right cosets Hx in G_n—this action of $gH \equiv \overline{g}$ induces the map

$$\overline{g}_* : \operatorname{Map}_c(H\backslash G_n, E_*) \to \operatorname{Map}_c(H\backslash G_n, E_*)$$

given by

(3.2) $$\overline{g}_*(f)(Hx) = f(Hg^{-1}x).$$

We can now state the main results of this paper. To make the notation less cumbersome, we will write $E(G)$ for E^{hG} when this object appears as a sub or superscript.

Theorem 3.1. *Let X be an E^{hK}-module. The $K(n)_*$-local E^{hH}-Adams spectral sequence $E_r^{s,t}(X, E^{hK}; E^{hH})$ in $\mathcal{D}_{E(K)}$ has E_2-term naturally isomorphic to $H_c^s(K/H, [X, E^{hH}]_t^{E(K)})$.*

Remark 3.2. By $K(n)_*$-local, we really mean $(K(n) \wedge E^{hK})^{E(K)}_*$-local, but, by the discussion in §1, the functors $L_{K(n)}$ and $L^{E(K)}_{K(n) \wedge E(K)}$ are naturally equivalent when restricted to E^{hK}-modules.

By Proposition 2.3, the next result implies that this spectral sequence converges strongly to $[X, E^{hH}_n]^{E(K)}_*$.

Theorem 3.3. *Every E^{hK}-module is $K(n)_*$-local E^{hH}-nilpotent in $\mathcal{D}_{E(K)}$.*

Corollary 3.4. *There is a strongly convergent spectral sequence*

$$H^s_c(K/H, (E^{hH})^*Z) \Rightarrow (E^{hK})^{t+s}(Z),$$

valid for any object Z in the stable category.

Proof. If Z is a cell S^0-module, apply Theorem 3.1 to $X = E^{hK} \wedge Z$ and use the fact that

$$[X, M]^{E(K)} = [Z, M]$$

whenever M is an E^{hK}-module. $\qquad\square$

To prove Theorem 3.1, we must first make sense of $H^s_c(K/H, [X, E^{hH}]^{E(K)}_t)$. Since K/H is a p-analytic profinite group [7, Theorem 9.6], $H^s_c(K/H, M)$ makes good sense whenever M is a profinite continuous $\mathbb{Z}_p[[K/H]]$-module: it can be defined as $\varprojlim_\alpha H^s_c(K/H, M_\alpha)$, where $M = \varprojlim_\alpha M_\alpha$ with each M_α a finite discrete K/H-module (see [6, Remark 1.3]). The action of K/H on E^{hH} induces an action on $[X, E^{hH}]^{E(K)}_t$; we thus need to provide $[X, E^{hH}]^{E(K)}_t$ with a natural topology so that it becomes a profinite continuous $\mathbb{Z}_p[[K/H]]$-module.

If I is the multi-index $(i_0, \ldots, i_{n-1})$, let $M(I)$ denote a finite spectrum with

$$BP_* M(I) = BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}),$$

provided such a spectrum exists.

Lemma 3.5. *Let X be a finite cell E^{hK}-module. Then $[X, E^{hH} \wedge M(I)]^{E(K)}_*$ is a finite discrete K/H-module in each degree.*

Proof. It suffices to show that $[X, E^{hH} \wedge M(I)]^{E(K)}_*$ is a finite discrete K/H-module in each degree when X is a "sphere E^{hK}-module" ([8, III,2]). Hence we need only show that $\pi_*(E^{hH} \wedge M(I))$ is a finite discrete K/H-module in each degree.

Consider the continuous homotopy fixed point spectral sequence

$$E^{*,*}_2 = H^*_c(H, E_*/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) \Rightarrow \pi_*(E^{hH} \wedge M(I))$$

from [6, Theorem 2]. Since $E_*/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$ is finite in each degree, it follows that $H^*_c(H, E_*/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))$ is finite in each bidegree (see e.g. [6, proof of Lemma 4.21]. But $E^{*,*}_\infty$ has a horizontal vanishing line since E^{hH} is $K(n)_*$-local E nilpotent (in the ordinary stable category) [6, Proposition A.3], and this spectral sequence is the $K(n)_*$-local E-Adams spectral sequence. This allows us to conclude that $\pi_*(E^{hH} \wedge M(I))$ is finite in each degree.

Furthermore,

$$\pi_*(E^{hH} \wedge M(I)) = \varinjlim_i \pi_*(E^{hHU_i} \wedge M(I)),$$

where the U_i's are as in (3.1). But $\pi_*(E^{hHU_i} \wedge M(I))$ is a discrete K/H module, since $K \cap U_i$ acts trivially on E^{hHU_i}. Hence $\pi_*(E^{hH} \wedge M(I))$ is a discrete K/H-module, and the proof is complete. $\square$

Proposition 3.6. *If X is any cell E^{hK}-module, then*

$$[X, E^{hH}]_*^{E(K)} = \lim_{\leftarrow \alpha, I}[X_\alpha, E^{hH} \wedge M(I)]_*^{E(K)},$$

where the X_α range over the finite E^{hK}-cell subcomplexes of X, and the $M(I)$ range over a sequence of multi-indices such that $\widehat{L}S^0 = \operatorname*{holim}_{\leftarrow I} M(I)$. With the topology of this inverse limit of discrete spaces, $[X, E^{hH}]_t^{E(K)}$ is therefore a profinite continuous K/H-module.

Proof. Since E^{hH} is $K(n)_*$-local,

$$E^{hH} \xrightarrow{\approx} \widehat{L}E^{hH} = \operatorname*{holim}_{\leftarrow I} E^{hH} \wedge M(I).$$

It therefore follows from the previous lemma that

$$[X_\alpha, E^{hH}]_*^{E(K)} = \lim_{\leftarrow I}[X_\alpha, E^{hH} \wedge M(I)]_*^{E(K)}$$

and in particular that $[X_\alpha, E^{hH}]_*^{E(K)}$ is a profinite group in each degree. Hence

$$\begin{aligned}
[X, E^{hH}]_*^{E(K)} &= \lim_{\leftarrow \alpha}[X_\alpha, E^{hH}]_*^{E(K)} \\
&= \lim_{\leftarrow \alpha, I}[X_\alpha, E^{hH} \wedge M(I)]_*^{E(K)},
\end{aligned}$$

completing the proof. $\square$

Now consider the E^{hH}-injective resolution $* \to E^{hK} \xrightarrow{\eta} I^0 \xrightarrow{d} I^1 \xrightarrow{d} \cdots$ in $\widehat{L}\mathcal{D}_{E(K)}$ given by

$$(3.3) \qquad\qquad I^j = \widehat{L}(\underbrace{E^{hH} \triangle_{E(K)} \cdots \triangle_{E(K)} E^{hH}}_{j+1 \text{ copies}})$$

with $\eta : E^{hK} \to E^{hH}$ the inclusion and $d : I^j \to I^{j+1}$ given by $d = \sum_{i=0}^{j+1}(-1)^i d^i$, where

$$d^i = \widehat{L}((E^{hH})^{(i)} \triangle_{E(K)} \eta \triangle_{E(K)} (E^{hH})^{(j+1-i)}).$$

(Recall from §1 that $\triangle_{E(K)}$ denotes the smash product in $\mathcal{D}_{E(K)}$.) Since iterated smash products will be ubiquitous, let us write

$$\Gamma^j(A; R) \equiv \underbrace{A \triangle_R A \triangle_R \cdots \triangle_R A}_{j+1 \text{ copies}}$$

whenever A is a commutative R-algebra (and R is a commutative S^0-algebra). Thus $\widehat{L}\Gamma^j(E^{hH}; E^{hK}) = I^j$.

Theorem 3.1 will follow easily from the next result.

Theorem 3.7. *Let X be a cell E^{hK}-module, and let*

$$T_j : [X, I^j]_*^{E(K)} \to \operatorname{Map}_c(\underbrace{K/H \times \cdots \times K/H}_{j \text{ copies}}, [X, E^{hH}]_*^{E(K)})$$

be defined by taking $T_j(f)(\overline{g}_1, \ldots, \overline{g}_j)$ to be the composition

$$X \xrightarrow{f} I^j \xrightarrow{\overline{g}_1^{-1} \wedge \cdots \wedge \overline{g}_j^{-1} \wedge E^{hH}} I^j \xrightarrow{\mu} E^{hH},$$

where we write $\overline{g} : E^{hH} \to E^{hH}$ for the action of $\overline{g} \in K/H$ on E^{hH}. Then T_j is an isomorphism.

Remark 3.8. It is not obvious that $T_j(f)$ is even a continuous map from $(K/H)^j$ to $[X, E^{hH}]_*^{E(K)}$. This will be proven along with the theorem.

Assuming Theorem 3.7, we can now prove half of our main result.

Proof of Theorem 3.1. With T_j as in 3.7, we have
(3.4)
$$T_{j+1}(df)(\overline{g}_1, \ldots, \overline{g}_{j+1}) = \sum_{i=0}^{j}(-1)^i T_j(f)(\overline{g}_1, \ldots, \widehat{\overline{g}}_{i+1}, \ldots, \overline{g}_{j+1})$$

$$+ (-1)^{j+1}\overline{g}_{j+1}^{-1}[T_j(f)(\overline{g}_1\overline{g}_{j+1}^{-1}, \ldots, \overline{g}_j\overline{g}_{j+1}^{-1})]$$

for all $f \in I^j$.

Define, for each discrete K/H-module M, a cochain complex $C^*(K/H, M)$ by

$$C^j(K/H, M) = \mathrm{Map}_c((K/H)^j, M)$$

with differential $d : C^j(K/H, M) \to C^{j+1}(K/H, M)$ given by

$$(df)(\overline{g}_1, \ldots, \overline{g}_{j+1}) = \sum_{i=0}^{j}(-1)^i f(\overline{g}_1, \ldots, \widehat{\overline{g}}_{i+1}, \ldots, \overline{g}_{j+1})$$

$$+ (-1)^{j+1}\overline{g}_{j+1}^{-1}[f(\overline{g}_1\overline{g}_{j+1}^{-1}, \ldots, \overline{g}_j\overline{g}_{j+1}^{-1})].$$

Then $H^*(C^*(K/H, ?))$ defines an effaceable δ-functor, and, since $H^0(C^*(K/H, M)) = M^{K/H}$, it follows that

$$H^*(C^*(K/H, M)) = H_c^*(K/H, M).$$

Of course, $C^*(K/H, M)$ can also be defined if $M = \varprojlim_{\alpha} M_\alpha$ is a profinite continuous $\mathbb{Z}_p[[K/H]]$-module, and the same proof as in [6, Lemma 4.21] applies to show that

$$
\begin{aligned}
H^*(C^*(K/H, M)) &= \varprojlim_{\leftarrow \alpha} H^*(C^*(K/H, M_\alpha)) \\
&= \varprojlim_{\leftarrow \alpha} H_c^*(K/H, M_\alpha) \\
&= H_c^*(K/H, M).
\end{aligned}
$$

Since 3.4 identifies the cochain complex $[X, I^*]_*^{E(K)}$ with $C^*(K/H, [X, E^{hH}]_*^{E(K)})$, it therefore follows that

$$E_2^{s,t}(X, E^{hK}; E^{hH}) = H_c^s(K/H, [X, E^{hH}]_*^{E(K)}),$$

as desired. □

Another consequence of Theorem 3.7 is the identification of the ring of homology cooperations (in the $K(n)_*$-local category) of E^{hH} regarded as an E^{hK}-algebra.

Corollary 3.9. *Let K and H be closed subgroups of G_n with H normal in K. Then*

$$\pi_* \widehat{L}(E^{hH} \triangle_{E(K)} E^{hH}) = \mathrm{Map}_c(K/H, E_*^{hH}).$$

The action $\overline{g}_*^L$ of $\overline{g} \in K/H$ on the left factor of $\pi_* \widehat{L}(E^{hH} \triangle_{E(K)} E^{hH})$ is given by

$$\overline{g}_*^L(f)(\overline{x}) = f(\overline{g}^{-1}\overline{x}),$$

and the action $\overline{g}_*^R$ of $\overline{g}$ on the right factor of $\pi_* \widehat{L}(E^{hH} \triangle_{E(K)} E^{hH})$ is given by

$$\overline{g}_*^R(f)(\overline{x}) = \overline{g}_* f(\overline{x}\,\overline{g}),$$

for any $f \in \mathrm{Map}_c(K/H, E_*^{hH})$.

The rest of this section is devoted to proving Theorem 3.7. We begin by computing $\pi_* \widehat{L}(I^j \triangle E)$. (Note here that, in taking the derived smash product with E, I^j is regarded as an object in $\mathcal{D}_{S^0}$.) This will require a number of lemmas; we first identify $\widehat{L}(I^j \triangle E)$ with $\widehat{L}[\Gamma^j(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E))]$.

Remark 3.10. The proof of Theorem 3.7 will be seen to reduce to the case where H and K are open in G_n, and thus we only need to understand $\widehat{L}(I^j \triangle E)$ in this case. However, we will carry out our analysis of $\widehat{L}(I^j \triangle E)$ in the general case, since some of our work along the way will be used in the next section.

Lemma 3.11. *There is a canonical equivalence $I^j \triangle E \simeq \Gamma^j(E^{hH} \wedge E; E^{hK} \wedge E)$ in $\mathcal{D}_{S^0}$.*

Proof. This result would be essentially immediate if derived smash products were not involved; their presence, however, forces us to do a little more work.

Replace E^{hH} by a weakly equivalent q-fibrant commutative E^{hK}-algebra $\overline{E}^{hH}$. Then, since E^{hK} is a q-cofibrant commutative S^0-algebra, it follows by Theorems 1.2 and 1.3 that I^j is represented in $\mathcal{D}_{E^{hK}}$ by (the ordinary smash product) $\overline{E}^{hH} \wedge_{E(K)} \cdots \wedge_{E(K)} \overline{E}^{hH}$. $\overline{E}^{hH} \wedge_{E(K)} \cdots \wedge_{E(K)} \overline{E}^{hH}$ is the coproduct of $j+1$ copies of $\overline{E}^{hH}$ in the category of commutative E^{hK}-algebras and is therefore q-cofibrant as well. Again using the fact that E^{hK} is a q-cofibrant commutative S^0-algebra, we conclude that $\overline{E}^{hH} \wedge_{E(K)} \cdots \wedge_{E(K)} \overline{E}^{hH}$ is also a q-cofibrant commutative S^0-algebra. Since E is a q-cofibrant commutative S^0-algebra, $I^j \triangle E$ is thus represented by $(\overline{E}^{hH} \wedge_{E(K)} \cdots \wedge_{E(K)} \overline{E}^{hH}) \wedge E$ in $\mathcal{D}_{S^0}$. But

$$(\overline{E}^{hH} \wedge_{E(K)} \cdots \wedge_{E(K)} \overline{E}^{hH}) \wedge E = (\overline{E}^{hH} \wedge_{E(K)} \cdots \wedge_{E(K)} \overline{E}^{hH}) \wedge (E \wedge_E \cdots \wedge_E E)$$
$$= (\overline{E}^{hH} \wedge E) \wedge_{E(K) \wedge E} \cdots \wedge_{E(K) \wedge E} (\overline{E}^{hH} \wedge E),$$

and $\overline{E}^{hH} \wedge E$ is a q-cofibrant commutative algebra over the q-cofibrant commutative S^0-algebra $E(K) \wedge E$. Therefore, $(\overline{E}^{hH} \wedge_{E(K)} \cdots \wedge_{E(K)} \overline{E}^{hH}) \wedge E$ represents $\Gamma^j(E^{hH} \wedge E; E^{hK} \wedge E)$ in $\mathcal{D}_{S^0}$, completing the proof. $\square$

Lemma 3.12. *The canonical map*

$$\Gamma^j(E^{hH} \wedge E; E^{hK} \wedge E) \to \Gamma^j(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E))$$

is a $K(n)_*$-equivalence in $\mathcal{D}_{S^0}$. Therefore,

$$\widehat{L}(I^j \triangle E) \simeq \widehat{L}[\Gamma^j(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E))].$$

Proof. First recall from §1 that $E^{hH} \wedge E$ and $E^{hK} \wedge E$ are cell commutative S^0-algebras, so that $\widehat{L}(E^{hH} \wedge E)$ and $\widehat{L}(E^{hK} \wedge E)$ are cell commutative S^0-algebras,

and $\widehat{L}(E^{hH} \wedge E)$ is a commutative $\widehat{L}(E^{hK} \wedge E)$-algebra. In addition, $\widehat{L}(E^{hH} \wedge E)$ is a commutative $(E^{hH} \wedge E)$-algebra, so we have the factorization

$$\Gamma^j(E^{hH} \wedge E; E^{hK} \wedge E) \xrightarrow{v} \Gamma^j(\widehat{L}(E^{hH} \wedge E); E^{hK} \wedge E)$$

$$\xrightarrow{w} \Gamma^j(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E)).$$

$K(n)$ is Bousfield equivalent to $K(n)^{(j+1)}$; we will therefore first prove that v is a $K(n)_*$-equivalence by proving that it is a $K(n)_*^{(j+1)}$-equivalence. Now

$$\Gamma^j(E^{hH} \wedge E; E^{hK} \wedge E) \wedge K(n)^{(j+1)} = \Gamma^j(E^{hH} \wedge E \wedge K(n); E^{hK} \wedge E)$$

with a similar statement for $\Gamma^j(\widehat{L}(E^{hH} \wedge E); E^{hK} \wedge E)$. (We are of course assuming that $K(n)$ is a cell S^0-module.) Since $E^{hH} \wedge E \wedge K(n) \xrightarrow{\cong} \widehat{L}(E^{hH} \wedge E) \wedge K(n)$, it is now immediate that v is a $K(n)_*^{(j+1)}$-equivalence.

That w is a $K(n)_*$-equivalence follows by induction from the next result, completing the proof. $\qquad\square$

Lemma 3.13. *Let F be a cell S^0-module, let R be a cell commutative S^0-algebra, and let LR be the F_*-localization of R (constructed to be an R-algebra and a cell commutative S^0-algebra.) If M and N are any LR-modules, then the canonical map $M\underline{\triangle}_R N \to M\underline{\triangle}_{LR} N$ is an F_*-equivalence.*

Proof. We will prove that

$$M\underline{\triangle}_R(N \wedge F) \to M\underline{\triangle}_{LR}(N \wedge F)$$

is an equivalence; this is equivalent to the conclusion of the lemma.

Let $\Gamma M \to M$ (resp. $\Gamma N \to N$) be a weak equivalence of LR-modules, where ΓM (resp. ΓN) is a cell LR-module, and let $\Gamma'M \to \Gamma M$ (resp. $\Gamma'N \to \Gamma N$) be a weak equivalence of R-modules, where $\Gamma'M$ (resp. $\Gamma'N$) is a cell R-module. Then, according to [8, IV, 7.5], we have a commutative diagram

$$
\begin{array}{ccccc}
B(\Gamma'M, R, \Gamma'N \wedge F) & \longrightarrow & \Gamma'M \wedge_R (\Gamma'N \wedge F) & \simeq & M\underline{\triangle}_R(N \wedge F) \\
\downarrow & & \downarrow & & \downarrow \\
B(\Gamma M, LR, \Gamma N \wedge F) & \longrightarrow & \Gamma M \wedge_{LR} (\Gamma N \wedge F) & \simeq & M\underline{\triangle}_{LR}(N \wedge F),
\end{array}
$$

and the horizontal arrows are weak homotopy equivalences. Here $B(X, Y, Z)$ denotes the geometric realization of the simplicial S^0-module $B_*(X, Y, Z)$, the simplicial bar construction for a right Y-module X and a left Y-module Z. Thus we need only show that the left vertical arrow is also a weak equivalence.

Since R and LR are q-cofibrant S^0-algebras, $B_*(\Gamma'M, R, \Gamma'N \wedge F)$ and $B_*(\Gamma M, LR, \Gamma N \wedge F)$ are proper simplicial S^0-modules [8, IV, 7.6], and it suffices to show that

(3.5) $$B_q(\Gamma'M, R, \Gamma'N \wedge F) \longrightarrow B_q(\Gamma M, LR, \Gamma N \wedge F)$$

$$\begin{array}{ccc}
\| & & \| \\
\Gamma'M \wedge R^{(q)} \wedge \Gamma'N \wedge F & \longrightarrow & \Gamma M \wedge (LR)^{(q)} \wedge \Gamma N \wedge F
\end{array}$$

is a weak equivalence. The map

$$\Gamma'M\underline{\triangle}R\underline{\triangle}\cdots\underline{\triangle}R\underline{\triangle}(\Gamma'N \wedge F) \to \Gamma M\underline{\triangle}LR\underline{\triangle}\cdots\underline{\triangle}LR\underline{\triangle}(\Gamma N \wedge F)$$

is certainly a weak equivalence; we thus only need to show that these derived smash products are represented by the point-set level smash products of Lemma 3.5. By Theorem 1.2, we are reduced to verifying that $\Gamma'M, \Gamma M, R, LR, \Gamma'N \wedge F$, and $\Gamma N \wedge F$ are all in $\overline{\mathcal{E}}_{S^0}$. But R and LR are in $\overline{\mathcal{E}}_{S^0}$ because they are q-cofibrant commutative S^0-algebras, and the others are in $\overline{\mathcal{E}}_{S^0}$ because they are either cell R-modules or cell LR-modules (see Proposition 1.4). This completes the proof. $\qquad\square$

We will use a Künneth type spectral sequence to compute

$$\pi_* \widehat{L}[\Gamma^j(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E))].$$

This requires understanding the structure of $\pi_* \widehat{L}(E^{hH} \wedge E) = \mathrm{Map}_c(H \backslash G_n, E_*)$ as a module over $\pi_* \widehat{L}(E^{hK} \wedge E) = \mathrm{Map}_c(K \backslash G_n, E_*)$. It will be useful to proceed a little more generally.

Say that a commutative ring C is a *complete ring* if it is provided with a complete decreasing filtration $\{F^s C\}_{s \geq 0}$ by ideals such that $F^s C \cdot F^t C \subset F^{s+t} C$ for all s, t. $C/F^s C$ is given the discrete topology, and $C = \varprojlim_s C/F^s C$ is then given the topology of the inverse limit.

Lemma 3.14. *Let S and T be profinite sets, and let C be a complete commutative ring. Then there is a canonical isomorphism*

$$\mathrm{Map}_c(S, C) \widehat{\otimes}_C \mathrm{Map}_c(T, C) \to \mathrm{Map}_c(S \times T, C).$$

Proof. By definition,

$$\mathrm{Map}_c(S, C) \widehat{\otimes}_C \mathrm{Map}_c(T, C) = \varprojlim_j \mathrm{Map}_c(S, C) \otimes_C \mathrm{Map}_c(T, C)/F^j,$$

where

$$F^j = \sum_{s+t=j} \mathrm{Map}_c(S, F^s C) \otimes_C \mathrm{Map}_s(T, F^t C).$$

Moreover, if X is any profinite set,

$$\mathrm{Map}_c(X, C) = \varprojlim_s \mathrm{Map}_c(X, C/F^s C) = \varprojlim_s \mathrm{Map}_c(X, C)/\mathrm{Map}_c(X, F^s C).$$

It therefore suffices to show that the canonical map

$$\mathrm{Map}_c(S, C/F^s C) \otimes_C \mathrm{Map}_c(T, C/F^s C) \to \mathrm{Map}_c(S \times T, C/F^s C)$$

is an isomorphism. But this follows by reduction to the case where S and T are finite. $\qquad\square$

Lemma 3.15. *Let G be a profinite group, and let H and K be closed subgroups of G with H normal in K. If C is any complete commutative ring, then*

$$\mathrm{Map}_c(H \backslash G, C) \approx \mathrm{Map}_c(H \backslash K, C) \widehat{\otimes}_C \mathrm{Map}_c(K \backslash G, C)$$

as $\mathrm{Map}_c(K \backslash G, C)$-algebras.

Proof. Since the above lemma implies that

$$\mathrm{Map}_c(H \backslash K, C) \widehat{\otimes}_C \mathrm{Map}_c(K \backslash G, C) = \mathrm{Map}_c(H \backslash K \times K \backslash G, C),$$

it suffices to show that there exists a homeomorphism $H \backslash K \times K \backslash G \to H \backslash G$ such that the diagram

$$\begin{array}{ccc} H \backslash K \times K \backslash G & \longrightarrow & H \backslash G \\ & \searrow \qquad \swarrow & \\ & K \backslash G & \end{array}$$

commutes. By [17, I §1, Proposition 1], there exists a continuous section $s : K\backslash G \to H\backslash G$. Then define $H\backslash K \times K\backslash G \to H\backslash G$ by sending (Hx, Kg) to $Hx \cdot s(Kg)$. $\square$

We can now make the desired computation of $\pi_*\widehat{L}[\Gamma^j(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E))]$.

Proposition 3.16. *Let $M(I)$ be a generalized Toda $V(n-1)$ corresponding to the multi-index $(i_0, \ldots, i_{n-1})$ as in the statement of Lemma 3.5, and let H and K be closed subgroups of G_n with H normal in K. Write $A = \pi_*\widehat{L}(E^{hK} \wedge E)$, $B = \pi_*\widehat{L}(E^{hH} \wedge E)$, and $J = (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}) \subset E_*$. Then*

$$B \otimes_A \cdots \otimes_A B / J(B \otimes_A \cdots \otimes_A B) \xrightarrow{\approx} \pi_*[\Gamma^j(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E)) \wedge M(I)].$$

Proof. We will prove this by induction on j, the result being trivial for $j = 0$. Begin by assuming that

$$\underbrace{B\widehat{\otimes}_A \cdots \widehat{\otimes}_A B}_{j \text{ copies}} \xrightarrow{\approx} \pi_*\widehat{L}[\Gamma^{j-1}(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E))],$$

where the completed tensor product is with respect to the J-adic filtration on B. Now

$$B\widehat{\otimes}_A \cdots \widehat{\otimes}_A B = \text{Map}_c(\underbrace{H\backslash G_n \times_{K\backslash G_n} \cdots \times_{K\backslash G_n} H\backslash G_n}_{j \text{ copies}}, E_*)$$

(cf. Lemma 3.14); in particular, $B\widehat{\otimes}_A \cdots \widehat{\otimes}_A B$ is flat over E_*.

Write $N = B\widehat{\otimes}_A \cdots \widehat{\otimes}_A B$. By [8, IV, Theorem 4.1], there is a spectral sequence

$$\text{Tor}_{s,t}^A(N, \pi_*[\widehat{L}(E^{hH} \wedge E) \wedge M(I)]) \Rightarrow \pi_{s+t}[\Gamma^j(\widehat{L}(E^{hH} \wedge E); \widehat{L}(E^{hK} \wedge E)) \wedge M(I)],$$

so we need only show that

$$\text{Tor}_{s,*}^A(N, \text{Map}_c(H\backslash G_n, E_*/J)) = 0$$

for all $s > 0$. N and A are both flat over E_*; therefore

$$\text{Tor}_{s,*}^A(N, \text{Map}_c(H\backslash G_n, E_*/J)) = \text{Tor}_{s,*}^{A/JA}(N/JN, \text{Map}_c(H\backslash G_n, E_*/J)).$$

Moreover, by the preceding lemma,

$$\text{Map}_c(H\backslash G_n, E_*/J) \approx \text{Map}_c(K/H, E_*/J) \otimes_{E_*/J} A/JA.$$

Since $\text{Map}_c(K/H, E_*/J)$ is flat over E_*/J, we then have

$$\begin{aligned} \text{Tor}_{s,*}^{A/JA}(N/JN, \text{Map}_c(H\backslash G_n, E_*/J)) &= \text{Tor}_{s,*}^{E_*/J}(N/JN, \text{Map}_c(K/H, E_*/J)) \\ &= 0 \end{aligned}$$

whenever $s > 0$. This completes the induction and the proof. $\square$

With the computation of $\pi_*\widehat{L}(I^j\underline{\wedge}E)$ in hand, we can now begin the proof of Theorem 3.7. We start by considering a special case; namely we assume that K/H is finite. Define an E^{hK}-module map

$$(3.6) \qquad \tau_j : I^j \to \bigvee_{(K/H)^j} E^{hH}$$

by requiring that the projection onto the summand indexed by $(\overline{g}_1, \ldots, \overline{g}_j)$ be given by the composition

$$I^j \xrightarrow{\widehat{L}(\overline{g}_1^{-1} \wedge \cdots \wedge \overline{g}_j^{-1} \wedge E^{hH})} I^j \xrightarrow{\mu} E^{hH}.$$

The next result implies immediately that T_j is an isomorphism in this case.

Proposition 3.17. *If K and H are closed subgroups of G_n with H normal in K and K/H finite, then τ_j is a weak equivalence.*

Proof. First assume that K and H are open in G_n. We will prove that

$$\widehat{L}(\tau_j \wedge E)_* : \pi_* \widehat{L}(I^j \underline{\triangle} E) \longrightarrow \pi_* \bigvee_{(K/H)^j} \widehat{L}(E^{hH} \wedge E)$$

$$\wr\wr$$

$$\bigoplus_{(K/H)^j} \mathrm{Map}(H\backslash G_n, E_*)$$

is an isomorphism. By Lemma 3.12 and Proposition 3.16,

$$\pi_* \widehat{L}(I^j \underline{\triangle} E) = \mathrm{Map}(H\backslash G_n, E_*) \otimes_{\mathrm{Map}(K\backslash G_n, E_*)} \cdots \otimes_{\mathrm{Map}(K\backslash G_n, E_*)} \mathrm{Map}(H\backslash G_n, E_*).$$

Tracking down the identifications, it follows from (3.2) that the projection $p_{(\overline{g}_1,\ldots,\overline{g}_j)}$ of $\widehat{L}(\tau_j \wedge E)_*$ onto the summand indexed by $(\overline{g}_1,\ldots,\overline{g}_j) \in (K/H)^j$ is given by

$$p_{(\overline{g}_1,\ldots,\overline{g}_j)}(f_1,\ldots,f_{j+1})(Hg) = f_1(\overline{g}_1 g)f_2(\overline{g}_2 g)\cdots f_j(\overline{g}_j g)f_{j+1}(Hg).$$

This formula also defines a map

$$z_j : \mathrm{Map}(H\backslash G_n, \mathbb{Z}_p) \otimes_{\mathrm{Map}(K\backslash G_n, \mathbb{Z}_p)} \cdots \otimes_{\mathrm{Map}(K\backslash G_n, \mathbb{Z}_p)} \mathrm{Map}(H\backslash G_n, \mathbb{Z}_p)$$

$$\rightarrow \bigoplus_{(K/H)^j} \mathrm{Map}(H\backslash G_n, \mathbb{Z}_p)$$

and $z_j \otimes E_* = \widehat{L}(\tau_j \wedge E)_*$. z_j is easily seen to be an epimorphism; we claim that this implies that z_j is in fact an isomorphism. Indeed, by Lemma 3.15,

$$\mathrm{Map}(H\backslash G_n, \mathbb{Z}_p) = \mathrm{Map}(K/H, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{Map}(K\backslash G_n, \mathbb{Z}_p),$$

so z_j becomes a map

$$\underbrace{\mathrm{Map}(K/H, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} \mathrm{Map}(K/H, \mathbb{Z}_p)}_{j+1 \text{ copies}} \otimes_{\mathbb{Z}_p} \mathrm{Map}(K\backslash G_n, \mathbb{Z}_p)$$

$$\rightarrow \bigoplus_{(K/H)^j} \mathrm{Map}(H\backslash G_n, \mathbb{Z}_p).$$

Both sides are free $\mathbb{Z}_p$-modules of the same finite rank and therefore z_j must be a monomorphism as well.

For the general case, recall the sequence $\{U_i\}$ of open normal subgroups of (3.1), and let $H_i = HU_i$, $K_i = KU_i$. Then

$$(3.7) \qquad \widehat{L}(\mathrm{holim}_{\rightarrow i} \Gamma^j(E^{hH_i}; E^{hK_i})) \xrightarrow{\cong} \widehat{L}(\Gamma^j(E^{hH}; E^{hK})).$$

There are several ways to see this. The most painless is to use Proposition 3.16 to verify that

$$\pi_* \widehat{L}(\mathrm{holim}_{\rightarrow i} \Gamma^j(E^{hH_i}; E^{hK_i})\underline{\triangle}E) \xrightarrow{\cong} \pi_* \widehat{L}(\Gamma^j(E^{hH}; E^{hK})\underline{\triangle}E).$$

Alternatively, one can use techniques such as Proposition 1.4 to replace derived smash products by ordinary smash products and then use their good properties with respect to colimits. Lemma 3.13 will also be involved at the end.

We have already proved that

$$\tau_j : \widehat{L}\Gamma^j(E^{hH_i}; E^{hK_i}) \rightarrow \bigvee_{(K_i/H_i)^j} E^{hH_i}$$

is an equivalence; the result in general now follows from (3.7), the naturality of τ_j, and the fact that $K/H \to K_i/H_i$ is a bijection for i sufficiently large. $\qquad\square$

Proof of Theorem 3.7. First observe that it suffices to show that

$$(3.8)\qquad T_j : [X_\alpha, I^j \wedge M(I)]_*^{E(K)} \to \mathrm{Map}_c((K/H)^j, [X_\alpha, E^{hH} \wedge M(I)]_*^{E(K)})$$

is an isomorphism for any finite cell E^{hK}-module X_α and multi-index I. Then, since $[X_\alpha, E^{hH} \wedge M(I)]_*^{E(K)}$ is finite in each degree (Lemma 3.5) and $\mathrm{Map}_c((K/H)^j, ?)$ is exact on the category of profinite abelian groups (see [17, I§1, Proposition 1]), all the relevant $\varprojlim^i$ terms vanish, and we have

$$[X, I^j]_*^{E(K)} \;=\; \varprojlim[X_\alpha, I^j \wedge M(I)]_*^{E(K)},$$

$$\mathrm{Map}_c((K/H)^j, [X, E^{hH}]_*^{E(K)}) \;=\; \varprojlim \mathrm{Map}_c((K/H)^j, [X_\alpha, E^{hH} \wedge M(I)]_*^{E(K)}).$$

Let $\{U_i\}$ be as in the previous proof, and let $H_i = K \cap HU_i$. Then K/H_i is finite, so from above,

$$T_j : [X_\alpha, \Gamma^j(E^{hH_i}; E^{hK}) \wedge M(I)]_*^{E(K)} \xrightarrow{\approx} \mathrm{Map}((K/H_i)^j, [X_\alpha, E^{hH_i} \wedge M(I)]_*^{E(K)}).$$

But, as in the proof above,

$$\varinjlim_{\to i} \Gamma^j(E^{hH_i}; E^{hK}) \wedge M(I) \to \Gamma^j(E^{hH}; E^{hK}) \wedge M(I) = I^j \wedge M(I)$$

is an equivalence in $\mathcal{D}_{E(K)}$; thus

$$[X_\alpha, I^j \wedge M(I)]_*^{E(K)} = \varinjlim_{\to i}[X_\alpha, \Gamma^j(E^{hH_i}; E^{hK}) \wedge M(I)]_*^{E(K)}.$$

Moreover,

$$\mathrm{Map}_c((K/H)^j, [X_\alpha, E^{hH} \wedge M(I)]_*^{E(K)}) = \varinjlim_{\to i} \mathrm{Map}((K/H_i)^j, [X_\alpha, E^{hH_i} \wedge M(I)]_*^{E(K)}),$$

and again it follows from the naturality of T_j that the map in (3.8) is an isomorphism. $\qquad\square$

4. CONVERGENCE

Our proof of Theorem 3.3 will involve two main steps. We will first prove that E^{hK} is $\widehat{L} - E_*^{E(K)}$ local. We will then establish a vanishing line in the E-Adams spectral sequence in $\widehat{L}\mathcal{D}_{E(K)}$; by virtue of Proposition 2.5, this implies that E^{hK} is E-nilpotent in $\widehat{L}\mathcal{D}_{E(K)}$. But E is an E^{hH}-module in $\mathcal{D}_{E(K)}$; therefore E is E^{hH}-nilpotent in $\widehat{L}\mathcal{D}_{E(K)}$, and hence so is E^{hK}. This immediately implies the conclusion of Theorem 3.3.

The proof of the first step requires an analysis of $\widehat{L}(E \wedge E^{hK})$. Recall that E and E^{hK} (and hence $E^{hK} \wedge E$ and $E \wedge E$) are cell commutative S^0-algebras; thus the maps

$$E^{hK} \;=\; E^{hK} \wedge S^0 \to E^{hK} \wedge E \to \widehat{L}(E^{hK} \wedge E),$$
$$E^{hK} \;=\; E^{hK} \wedge S^0 \to E \wedge E \to \widehat{L}(E \wedge E)$$

are algebra maps. In particular, these maps give $\widehat{L}(E^{hK} \wedge E)$ and $\widehat{L}(E \wedge E)$ the structure of E^{hK}-modules.

Lemma 4.1. $\widehat{L}(E^{hK} \wedge E)$ *is a retract of* $\widehat{L}(E \wedge E)$ *in* $\mathcal{D}_{E(K)}$.

Proof. We will prove that $\widehat{L}(E^{hK} \wedge E)$ is a retract of $\widehat{L}(E \wedge E)$ in $\mathcal{D}_{\widehat{L}(E(K) \wedge E)}$; that is, there exists a map $r : \widehat{L}(E \wedge E) \to \widehat{L}(E^{hK} \wedge E)$ in $\mathcal{D}_{\widehat{L}(E(K) \wedge E)}$ such that the composition

$$\widehat{L}(E^{hK} \wedge E) \to \widehat{L}(E \wedge E) \xrightarrow{r} \widehat{L}(E^{hK} \wedge E)$$

is a weak equivalence. The desired result then follows upon applying the functor $\mathcal{D}_{\widehat{L}(E(K) \wedge E)} \to \mathcal{D}_{E(K)}$.

First observe that there is a retraction

$$h : \pi_* \widehat{L}(E \wedge E) \to \pi_* \widehat{L}(E^{hK} \wedge E)$$

of $\pi_* \widehat{L}(E^{hK} \wedge E)$ modules. Indeed, let h be the map

$$\pi_* \widehat{L}(E \wedge E) = \mathrm{Map}_c(G_n, E_*) \xrightarrow{\mathrm{Map}_c(s, E_*)} \mathrm{Map}_c(K \backslash G_n, E_*) = \pi_* \widehat{L}(E^{hK} \wedge E),$$

where $s : K \backslash G_n \to G_n$ is a continuous cross-section of the projection $G_n \to K \backslash G_n$ ([17, I§1, Proposition 1]). It is clear that h is a retraction; we thus only need to show that it is a map of $\pi_* \widehat{L}(E^{hK} \wedge E)$-modules. This amounts to showing that the diagram

$$
\begin{array}{ccc}
\mathrm{Map}_c(K \backslash G_n, E_*) \otimes \mathrm{Map}_c(G_n, E_*) & \xrightarrow{\mathrm{Map}_c(K \backslash G_n, E_*) \otimes h} & \mathrm{Map}_c(K \backslash G_n, E_*) \otimes \mathrm{Map}_c(K \backslash G_n, E_*) \\
\downarrow & & \downarrow \\
\mathrm{Map}_c(K \backslash G_n \times G_n, E_*) & \xrightarrow{\mathrm{Map}_c(K \backslash G_n \times s, E_*)} & \mathrm{Map}_c(K \backslash G_n \times K \backslash G_n, E_*) \\
\downarrow & & \downarrow \\
\mathrm{Map}_c(G_n, E_*) & \xrightarrow{\mathrm{Map}_c(s, E_*)} & \mathrm{Map}_c(K \backslash G_n, E_*)
\end{array}
$$

commutes, where the lower vertical maps are induced by diagonal maps. But this is easy to check.

Now write $A = \pi_* \widehat{L}(E^{hK} \wedge E)$ and $B = \pi_* \widehat{L}(E \wedge E)$. There is a spectral sequence

$$\mathrm{Ext}_A^{**}(B, A/JA) \Rightarrow [\widehat{L}(E \wedge E), \widehat{L}(E^{hK} \wedge E \wedge M(I)]_*^{\widehat{L}(E(K) \wedge E)}$$

[8, IV, Theorem 4.1]. As usual, I is the multi-index $(i_0, \ldots, i_{n-1})$, and J is the ideal $(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ in E_*. We will show that $\mathrm{Ext}_A^{s,*}(B, A/JA) = 0$ for all $s > 0$; this implies that there is a retraction r in $\mathcal{D}_{\widehat{L}(E(K) \wedge E)}$ with $\pi_* r = h$.

Since B and A are both flat over E_*,

$$\mathrm{Ext}_A^{**}(B, A/JA) = \mathrm{Ext}_{A/JA}(B/JB, A/JA).$$

By Lemma 3.15 with $H = \{e\}$,

$$B/JB \approx \mathrm{Map}_c(K, E_*/J) \otimes_{E_*/J} A/JA$$

as A/JA-algebras. Since $\mathrm{Map}_c(K, E_*/J)$ is a flat E_*/J-module, it then follows that

$$\mathrm{Ext}_{A/JA}^{**}(B/JB, A/JA) = \mathrm{Ext}_{E_*/J}^{**}(\mathrm{Map}_c(K, E_*/J), A/JA)$$

and that

$$\mathrm{Ext}_{E_*/J}^{**}(\mathrm{Map}_c(K, E_*/J), N) = \mathrm{Ext}_{\mathbb{F}_{p^n}[u,u^{-1}]}^{**}(\mathrm{Map}_c(K, E_*/I_n), N)$$

whenever N is an E_*-module annihilated by the ideal $I_n = (p, v_1, \ldots, v_{n-1})$. Hence

$$\mathrm{Ext}_{E_*/J}^{s,*}(\mathrm{Map}_c(K, E_*/J), N) = 0$$

for all $s > 0$. Now consider the short exact sequences

$$0 \to \frac{(I_n^k + J)A}{(I_n^{k+1} + J)A} \to \frac{A}{(I_n^{k+1} + J)A} \to \frac{A}{(I_n^k + J)A} \to 0,$$

and take $N = (I_n^k + J)A/(I_n^{k+1} + J)A$. Using induction on k, we obtain that

$$\mathrm{Ext}_{E_*/J}^{s,*}(\mathrm{Map}_c(K, E_*/J), A/(I_n^k + J)A) = 0$$

for all $s > 0$ and $k \geq 0$. But $I_n^k \subset J$ for k sufficiently large; therefore

$$\mathrm{Ext}_A^{s,*}(B, A/JA) = \mathrm{Ext}_{E_*/J}^{s,*}(\mathrm{Map}_c(K, E_*/J), A/JA) = 0$$

whenever $s > 0$. $\qquad\square$

Lemma 4.2. E^{hK} is $\widehat{L} - E_*^{E(K)}$ local.

Proof. Let X be a cell E^{hK}-module. We must show that if $X \wedge_{E(K)} (E \wedge K(n)) \simeq *$ in $\mathcal{D}_{E(K)}$, then $[X, E^{hK}]_*^{E(K)} = 0$. As usual, we take $K(n)$ to be a cell S^0-module.

Let ΓE be a cell E^{hK}-module and $\Gamma E \to E$ a weak equivalence of E^{hK}-modules. Then $X \wedge_{E(K)} (E \wedge K(n))$ is trivial in $\mathcal{D}_{E(K)}$ if and only if $(X \wedge_{E(K)} \Gamma E) \wedge K(n)$ is contractible in $\mathcal{M}_{E(K)}$. This implies that $[X \wedge_{E(K)} (\Gamma E \wedge E)] \wedge K(n)$ is contractible, where E^{hK} acts on the left factor of $\Gamma E \wedge E$. But $\Gamma E \wedge E \to E \wedge E$ is a weak equivalence since E is a q-cofibrant commutative S^0-algebra and ΓE is a cell module over a q-cofibrant commutative S^0-algebra (see Theorem 1.2–Proposition 1.4). Hence $[X \wedge_{E(K)} (E \wedge E)] \wedge K(n)$, and therefore $[X \wedge_{E(K)} \widehat{L}(E \wedge E)] \wedge K(n)$, is weakly contractible. (Here we have applied Proposition 1.1 a number of times.) By the preceding lemma, it follows that $(X \wedge_{E(K)} \widehat{L}(E^{hK} \wedge E)) \wedge K(n)$ is also weakly contractible. Thus

$$[X \wedge_{E(K)} (E^{hK} \wedge E)] \wedge K(n) = X \wedge E \wedge K(n)$$

is weakly contractible. Now apply Propositions 1.1–1.4 again to obtain that $X \triangle E \triangle K(n)$ is weakly equivalent to $X \wedge E \wedge K(n)$; since $E \triangle K(n)$ is a wedge of $K(n)$'s, this implies that X is $K(n)_*$-acyclic. But E^{hK} is $K(n)_*$-local, and the $K(n)_*$-localization of an E^{hK}-module is the same as the $(E^{hK} \wedge K(n))_*^{E(K)}$-localization; therefore $[X, E^{hK}]_*^{E(K)} = 0$. This completes the proof. $\qquad\square$

The next result provides us with the requisite vanishing line.

Lemma 4.3. *Let K be a closed subgroup of G_n, let P be a p-Sylow subgroup of K, and let X be the p-localization of a finite spectrum with free $\mathbb{Z}_{(p)}$-homology such that*

$$H_c^{s,*}(P, E_*X/I_n E_*X) = 0$$

for all s bigger than some s_0. Then, if Z is any E^{hK}-module,

$$H_c^{s,*}(K, [Z, E \wedge X]_{E(K)}^*) = 0$$

for all $s > s_0$.

Remark 4.4. The topology on $[Z, E \wedge X]_{E(K)}^* = [Z \wedge DX, E]_{E(K)}^*$ is just that of Proposition 3.6 with $H = \{e\}$.

Once this lemma is proved, we easily obtain the following result, which, by the remarks at the beginning of this section, implies Theorem 3.3.

Lemma 4.5. E^{hK} *is E-nilpotent in $\widehat{L}\mathcal{D}_{E(K)}$.*

Proof. By the discussion in [6, proof of Theorem 5.3], there exists an X satisfying the hypotheses of the previous lemma. It then follows from Proposition 2.5 that $E^{hK} \wedge X$ is E-nilpotent in $\widehat{L}\mathcal{D}_{E(K)}$.

Now consider the class $\mathcal{N}$ consisting of all finite p-local spectra Y such that $E^{hK} \wedge Y$ is E-nilpotent in $\widehat{L}\mathcal{D}_{E(K)}$. Nilpotence technology [13, Theorem 7] tells us that $\mathcal{N}$ is the collection of $K(m-1)_*$-acyclic spectra for some $m \geq 0$. Since $X \in \mathcal{N}$, we must have that $\mathcal{N}$ consists of all finite p-local spectra. In particular, $S^0_{(p)} \in \mathcal{N}$ and thus E^{hK} is E-nilpotent in $\widehat{L}\mathcal{D}_{E(K)}$. $\qquad\square$

We now turn to the proof of Lemma 4.3. Say that an $\mathbb{F}_{p^n}$-vector space M is a *twisted K-module over $\mathbb{F}_{p^n}$* if it is a $\mathbb{Z}[K]$-module such that

$$g(cm) = (\pi(g)c) \cdot gm$$

for all $g \in K$, $c \in \mathbb{F}_{p^n}$, and $m \in M$, where $\pi : K \to \mathrm{Gal}$ is the usual projection. We then have the following preliminary result. Our notation is as in Lemma 4.3.

Lemma 4.6. *Suppose that N is a finite discrete twisted K-module over $\mathbb{F}_{p^n}$ with the property that $H^s_c(P, N) = 0$ for all s bigger than some s_0. Then $H^s_c(K, M \otimes_{\mathbb{F}_{p^n}} N) = 0$ for all $s > s_0$ whenever M is a finite discrete twisted K-module over $\mathbb{F}_{p^n}$ and K acts diagonally on $M \otimes_{\mathbb{F}_{p^n}} N$.*

Proof.
$$H^*_c(K, M \otimes_{\mathbb{F}_{p^n}} N) \to H^*_c(P, M \otimes_{\mathbb{F}_{p^n}} N)$$
is a monomorphism, so we need only show that $H^s_c(P, M \otimes_{\mathbb{F}_{p^n}} N) = 0$ whenever $s > s_0$.

Let L be the image of the projection $\pi : P \to \mathrm{Gal}$, so that $L = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^m})$ for some $m \geq 1$, and let P_0 be the kernel of π. Since P_0 is p-analytic, $H^j_c(P_0, M \otimes_{\mathbb{F}_{p^n}} N)$ is a finite-dimensional $\mathbb{F}_{p^n}$-vector space V_j. V_j is acted upon by L, and $\sigma(cv) = \sigma(c)\sigma(v)$ for all $\sigma \in L$, $c \in \mathbb{F}_{p^n}$, and $v \in V_j$. It then follows (cf. [5, Lemma 5.4]) that $V_j = V_j^L \otimes_{\mathbb{F}_{p^m}} \mathbb{F}_{p^n}$ as $\mathbb{F}_{p^n}$-vector spaces and as L-modules, so that

$$H^i(L, H^j_c(P_0, M \otimes_{\mathbb{F}_{p^n}} N)) = \left\{ \begin{array}{ll} H^j_c(P_0, M \otimes_{\mathbb{F}_{p^n}} N)^L & i = 0, \\ 0 & i \neq 0. \end{array} \right.$$

Therefore, by the Lyndon-Hochschild-Serre spectral sequence for profinite groups (see [18, II, §4]),

$$H^s_c(P, M \otimes_{\mathbb{F}_{p^n}} N) = 0 \Leftrightarrow H^s_c(P_0, M \otimes_{\mathbb{F}_{p^n}} N) = 0.$$

The point here is that M and N are both P_0-modules over $\mathbb{F}_{p^n}$; there is no Galois action on $\mathbb{F}_{p^n}$ to worry about.

Since P_0 is a profinite p-group, the only finite simple discrete P_0-module over $\mathbb{F}_{p^n}$ is $\mathbb{F}_{p^n}$ with the trivial action (cf. [18, Propositon 17]). By considering a composition series for M, it now follows easily that $H^s_c(P_0, M \otimes_{\mathbb{F}_{p^n}} N) = 0$ whenever $s > s_0$. This completes the proof. $\qquad\square$

Proof of Lemma 4.3. By Proposition 3.6 and the definition of cohomology, it suffices to prove that
$$H^{s,*}_c(K, [Z \wedge M(I), E \wedge X]^*_{E(K)}) = 0$$
for all $s > s_0$ and multi-indices I, whenever Z is a finite cell E^{hK}-module.

Consider now the map
$$E^*_{E(K)}(Z \wedge M(I)) \otimes_{E_*} E_* X \to [Z \wedge M(I), E \wedge X]^*_{E(K)}$$

which sends $f \otimes c$ to the composition

$$Z \wedge M(I) = Z \wedge M(I) \wedge S^0 \xrightarrow{f \wedge c} E \wedge E \wedge X \xrightarrow{\mu \wedge X} E \wedge X.$$

Using induction on the E^{hK}-cells of $Z \wedge M(I)$, together with the fact that $E_* X$ is a free E_*-module, it is easy to check that this map is an isomorphism. Moreover, $E^*_{E(K)}(Z \wedge M(I))$ is annihilated by a finite power of I_n; it therefore suffices to show that

$$H_c^{s,*}(K, I_n^k E^*_{E(K)}(Z \wedge M(I))/I_n^{k+1} E^*_{E(K)}(Z \wedge M(I)) \otimes_{E_*/I_n} E_* X/I_n E_* X) = 0$$

for all $s > s_0$ and $k \geq 0$. But this follows from Lemma 4.6. $\qquad \square$

Appendix A. A consistency result

Let R be a commutative S^0-algebra, and let X be an R-module on which a finite group G acts by R-module maps. This action defines an evident functor from the category with one object and morphism group G to the category of R-modules; we write $\Pi^* X$ for the cosimplicial replacement of this diagram. (Here, and in what follows, we use the terminology and notation of [4, Chapters X and XI].) $\mathrm{Tot}(\Pi^* X) = X^{hG}$, and the homotopy fixed point spectral sequence converging to $[Z, X^{hG}]^*_R$ is obtained by mapping Z into the tower of fibrations $\{\mathrm{Tot}_k \Pi^* X\}$. If K and H are closed subgroups of G_n with H normal in K and K/H finite, then we may specialize to the case where $R = E^{hK}$, $X = E^{hH}$, and $G = K/H$.

Theorem A.1. *With the notation as above, the homotopy fixed point spectral sequence agrees with the Adams spectral sequence $E_r^{*,*}(Z, E^{hK}; E^{hH})$ in $\widehat{L} \mathcal{D}_{E(K)}$.*

Proof. Let F_k denote the fiber of $\mathrm{Tot}_k(\Pi^* E^{hH}) \to \mathrm{Tot}_{k-1}(\Pi^* E^{hH})$. It suffices to prove that the sequence

$$* \to E^{hK} \to F_0 \to \Sigma F_1 \to \cdots$$

is an E^{hH}-Adams resolution of E^{hK} in $\widehat{L} \mathcal{D}_{E(K)}$. Each F_k is a product of suspensions of copies of E^{hH} and so is E^{hH}-injective in $\widehat{L} \mathcal{D}_{E(K)}$. We are therefore reduced to showing that

$$(A.1) \qquad 0 \to [Z, \widehat{L}(E^{hK} \triangle_{E(K)} E^{hH}]_*^{E(K)} \to [Z, \widehat{L}(F_0 \triangle_{E(K)} E^{hH}]_*^{E(K)} \to \cdots$$

is exact for all cell E^{hK}-modules Z. But $\widehat{L}(F_k \triangle_{E(K)} E^{hH})$ is equivalent to the fiber of

$$(A.2) \qquad \mathrm{Tot}_k(\Pi^* \widehat{L}(E^{hH} \triangle_{E(K)} E^{hH})) \to \mathrm{Tot}_{k-1}(\Pi^* \widehat{L}(E^{hH} \triangle_{E(K)} E^{hH})),$$

where K/H acts on the left factor of $\widehat{L}(E^{hH} \triangle_{E(K)} E^{hH})$. Hence

$$H^*[Z, \widehat{L}(F_* \triangle_{E(K)} E^{hH})]^*_{E(K)} = H^*(K/H, [Z, \widehat{L}(E^{hH} \triangle_{E(K)} E^{hH})]^*_{E(K)}).$$

By Corollary 3.9,

$$[Z, \widehat{L}(E^{hH} \triangle_{E(K)} E^{hH})]^*_{E(K)} = \mathrm{Map}_c(K/H, [Z, E^{hH}]^*_{E(K)})$$

as K/H-modules, and therefore

$$H^*(K/H, [Z, \widehat{L}(E^{hH} \triangle_{E(K)} E^{hH})]^i_{E(K)}) = [Z, E^{hH}]^i_{E(K)}$$

concentrated in homological degree 0. This proves that the desired sequence is exact. $\qquad \square$

Remark A.2. The alert reader may have noticed that $\widehat{L}(E^{hH}\triangle_{E(K)}E^{hH})$ in (A.2) is only a homotopy object and therefore the action of K/H is only an action up to homotopy. However, an honest action may be constructed by observing that $E^{hH}\triangle_{E(K)}E^{hH}$ is equivalent to $E^{hH} \wedge_{E(K)} \Gamma E^{hH}$, where ΓE^{hH} is a cell E^{hK}-module weakly equivalent to E^{hH}, and that $\widehat{L}$ can be constructed to be natural on $\mathcal{M}_{E(K)}$ (see [8, Chapter VIII]).

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